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Completeness of Bethe's states for the generalized *XXZ* model

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Dedicated to the memory of Ansgar Schnizer

Abstract. We study the Bethe ansatz equations for a generalized XXZ model on a one-dimensional lattice. Assuming the string conjecture we propose an integer version for vacancy numbers and prove a combinatorial completeness of Bethe's states for a generalized XXZ model. We find an exact form for the inverse matrix related with vacancy numbers and compute its determinant. This inverse matrix has a tridiagonal form, generalizing the Cartan matrix of type A.

1. Introduction

An integrable generalization of spin-$\frac{1}{2}$ Heisenberg XXZ model to arbitrary spins was given, for example, in [KR2]. As a matter of fact, a spectrum of the generalized XXZ model is described by the solutions $\{\lambda_i\}$ to the following system of equations ($1 \leqslant j \leqslant l$):

$$\prod_{a=1}^{N} \frac{\sinh \frac{1}{2}\theta(\lambda_j + 2\mathrm{i}s_a)}{\sinh \frac{1}{2}\theta(\lambda_j - 2\mathrm{i}s_a)} = \prod_{\substack{k=1\\k\neq j}}^{l} \frac{\sinh \frac{1}{2}\theta(\lambda_j - \lambda_k + 2\mathrm{i})}{\sinh \frac{1}{2}\theta(\lambda_j - \lambda_k - 2\mathrm{i})}. \tag{1.1}$$

Here θ is an anisotropy parameter, s_a, $1 \leqslant a \leqslant N$, are the spins of atoms in the magnetic chain and l is the number of magnons over the ferromagnetic vacuum.

The main goal of our paper is to present a computation the number of solutions to system (1.1) based on the so-called string conjecture (see, e.g., [TS], [KR1]). In spite of the well known fact that solutions of (1.1) do not have, in general, a 'string nature' (see, e.g., [EKK]), we prove that the string conjecture gives a correct answer for the number of solutions to the system of equations (1.1). Note that a combinatorial completeness of Bethe's states for the generalized XXX Heisenberg model was proved in [K1] and appears to be a starting point for numerous applications to combinatorics of Young tableaux and representation theory of symmetric and general linear groups (see, e.g., [K2]).

2. Analysis of the Bethe equations

Let us consider the XXZ model of spins $s_1, \ldots, s_k$ interacting on a one-dimensional lattice with the each spin s_i repeated N_i times. In the standard XXZ model all spins s_i are equal to $\frac{1}{2}$. Let Δ be the anisotropy parameter (see, e.g., [TS], [KR2]). We assume that $0 < \Delta < 1$.

Let us pick out a real number θ such that $\cos\theta = \Delta,\ 0 < \theta < \frac{\pi}{2}$, and denote

$$p_0 = \frac{\pi}{\theta} > 2.$$

Each spin s has a 'parity' v_{2s} which is equal to plus or minus one.

Bethe vectors $\psi(x_1, \ldots, x_l)$ for XXZ model are parametrized by l complex numbers $x_j (\mathrm{mod}\ 2p_0 \mathrm{i})$ $(l \leqslant 2s_1 N_1 + \cdots + 2s_k N_k)$, which satisfy the following system of transcendental equations (Bethe's equations),

$$\prod_{m=1}^{k} (-1)^{N_m v_{2s_m}} \left(\frac{\sinh \frac{1}{2}\theta(x_\alpha + \eta_m - \mathrm{i}(2s_m + \frac{1}{2}(1 - v_{2s_m}) p_0))}{\sinh \frac{1}{2}\theta(x_\alpha + \eta_m + \mathrm{i}(2s_m + \frac{1}{2}(1 - v_{2s_m}) p_0))} \right)^{N_m} = -\prod_{j=1}^{l} \frac{\sinh \frac{1}{2}\theta(x_\alpha - x_j - 2\,\mathrm{i})}{\sinh \frac{1}{2}\theta(x_\alpha - x_j + 2\,\mathrm{i})} \tag{2.1}$$

where $\alpha = 1, \ldots, l$ and $\{\eta_m\}$ are some fixed real numbers, and the non-degeneracy conditions, the norm of the Bethe's vectors ψ are not equal to zero, apply.

Solutions to the system (2.1) are considered modulo $2p_0 \mathrm{i}\mathbb{Z}$, because $\sinh(\frac{1}{2}\theta x)$ is a periodic function with the period $2p_0\mathrm{i}$. Asymptotically for $N_m \to \infty$, $1 \leqslant m \leqslant k$ and finite l the solutions to the system (2.1) create the strings. The strings are characterized by the common real abscissa, which is called the string centre, the length n and parity v_n. Centres of even strings are located on the line $\mathrm{Im}\, x = 0$ (and $v_n = +1$), those of odd strings are located on the line $\mathrm{Im}\, x = p_0$ (and $v_n = -1$). A string of length n and parity v_n consists of n complex numbers $x^n_{\beta,j}$ of the following form,

$$x^n_{\beta,j} = x^n_\beta + \mathrm{i}\left(n + 1 - 2j + \frac{1 - v_n}{2} p_0) + \mathrm{O}(\exp(-\delta N)\right)(\mathrm{mod}\ 2p_0\mathrm{i}) \tag{2.2}$$

where $\delta > 0$, $j = 1, \ldots, n$, $x^n_\beta \in \mathbb{R}$.

A distribution of numbers $\{x_j\}$ on strings is called a configuration. Each configuration can be parametrized by the filling numbers $\{\lambda_n\}$, where λ_n is equal to the number of strings with length n and parity v_n. Each real solution of the system (2.1) (modulo $2p_0\mathrm{i}$), corresponds to an even string of length 1. Configuration parameters $\{\lambda_n\}$, $n \geqslant 1$, satisfy the following conditions: $\lambda_n \geqslant 0$, $\sum_{n \geqslant 1} n\lambda_n = l$. The system (2.1) can be transformed into that for real numbers x^n_β for each fixed configuration. To get such a system, let us calculate the scattering phase $\theta_{n,m}(x)$ of the string length n on that of length m. By definition

$$\exp(-2\pi \mathrm{i}\theta_{n,m}(x)) = \prod_{j=1}^{n}\prod_{k=1}^{m} \frac{\sinh \frac{1}{2}\theta(x^n_{\alpha,j} - x^m_{\beta,k} - 2\mathrm{i})}{\sinh \frac{1}{2}\theta(x^n_{\alpha,j} - x^m_{\beta,k} + 2\mathrm{i})} \qquad x := x^n_\alpha - x^m_\beta. \tag{2.3}$$

From the formulae

$$\mathrm{Im}\log\left(\frac{\sinh(\lambda + a\mathrm{i})}{\sinh(\mu + b\mathrm{i})}\right) = \arctan(\tanh\mu \cdot \cot b) - \arctan(\tanh\lambda \cdot \cot a) \tag{2.4}$$

where $a, b, \lambda, \mu \in \mathbb{R}$, it follows

$$\begin{aligned} -\pi\theta_{n,m}(x) &= \sum_{j=1}^{n}\sum_{k=1}^{m} \arctan(\tanh \tfrac{1}{2}\theta x \cot \tfrac{1}{2}\theta(n - m - 2j + 2k + \tfrac{1}{2}(v_m - v_n)p_0)) \\ &= \arctan(\tanh \tfrac{1}{2}\theta x \cdot \cot \tfrac{1}{2}\theta(m + n + \tfrac{1}{2}(1 - v_n v_m)p_0)) \\ &\quad + \arctan(\tanh \tfrac{1}{2}\theta x \cdot \cot \tfrac{1}{2}\theta(|n - m| + \tfrac{1}{2}(1 - v_n v_m)p_0)) \\ &\quad + 2\sum_{s=1}^{\min(n,m)-1} \arctan(\tanh \tfrac{1}{2}\theta x \cdot \cot \tfrac{1}{2}\theta(|n - m| + 2s + \tfrac{1}{2}(1 - v_n v_m)p_0)). \end{aligned}$$

Now let us consider the limit of $\theta_{n,m}(x)$ when $x \to \infty$. Note that for $x \to \infty$, we have $\tanh \frac{1}{2}\theta x \to 1$, and $\arctan(\cot z) = -\pi((z/\pi))$, if $z/\pi \notin \mathbb{Z}$, where $((z))$ is the Dedekind function:

$$((z)) = \begin{cases} 0 & \text{if } z \in \mathbb{Z} \\ \{z\} - \frac{1}{2} & \text{if } z \notin \mathbb{Z} \end{cases}$$

and $\{z\} = z - [z]$ is the fractional part of z. Then

$$\theta_{n,m}(\infty) = \left(\left(\frac{n+m}{2p_0} + \frac{1 - v_n v_m}{4}\right)\right) + \left(\left(\frac{|n-m|}{2p_0} + \frac{1 - v_n v_m}{4}\right)\right)$$
$$+2 \sum_{l=1}^{\min(n,m)-1} \left(\left(\frac{|n-m| + 2l}{2p_0} + \frac{1 - v_n v_m}{4}\right)\right).$$

Let us define

$$\Phi_{n,m}(\lambda) = -\frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{n} \log \frac{\sinh \frac{1}{2}\theta(x^n_{\alpha,j} + \eta - m\mathrm{i} - \frac{1}{2}(1 - v_m)p_0\mathrm{i})}{\sinh \frac{1}{2}\theta(x^n_{\alpha,j} + \eta + m\mathrm{i} + \frac{1}{2}(1 - v_m)p_0\mathrm{i})}$$
$$\lambda := x^n_\alpha + \eta \qquad \eta \in \mathbb{R}$$

then

$$\Phi_{n,m}(\lambda) = -\frac{1}{2\pi} \sum_{j=1}^{n} 2 \cdot \arctan(\tanh \tfrac{1}{2}\theta x \cot \tfrac{1}{2}\theta(n - m + 1 - 2j + \tfrac{1}{2}(v_m - v_n)p_0))$$
$$= \frac{1}{\pi} \sum_{l=1}^{\min(n,m)} \arctan(\tanh \tfrac{1}{2}\theta x \cot \tfrac{1}{2}\theta(|n-m| + 2l - 1 + \tfrac{1}{2}(1 - v_n v_m)p_0))$$

and, consequently,

$$\Phi_{n,m}(\infty) = \sum_{l=1}^{\min(n,m)} \left(\left(\frac{|n-m| + 2l - 1}{2p_0} + \frac{1 - v_n v_m}{4}\right)\right). \tag{2.5}$$

Now let us continue the investigation of system (2.1). Multiplying the equations of the system (2.1) along the string $x^n_{\alpha,j}$ and taking the logarithm result, one can obtain the following system on real numbers $x^n_\alpha, \alpha = 1, \ldots, \lambda_n$:

$$\sum_m \Phi_{n,2s_m}(x^n_\alpha + \eta_m)N_m = Q^n_\alpha + \sum_{(\beta,m)\neq(\alpha,n)} \theta_{n,m}(x^n_\alpha - x^m_\beta) \qquad \alpha = 1, \ldots, \lambda_n. \tag{2.6}$$

Integer or half-integer numbers Q^n_α, $1 \leqslant \alpha \leqslant \lambda_n$, are called quantum numbers. They parametrize—according to the string conjecture [TS], [FT], [K1]—the solutions to the system (2.1). Admissible values of quantum numbers Q^n_α are located in the symmetric interval $[-Q^n_{\max}, Q^n_{\max}]$ and appear to be an integer or half-integer in accordance with that of $Q^n_{\max}$.

3. Calculation of vacancy numbers

Following [TS], we will assume that there are two types of length 1 string, namely even and odd types. If the length n of a string is greater than 1, then n and parity v_n satisfy the following conditions:

$$v_n \cdot \sin(n-1)\theta > 0 \tag{3.1}$$
$$v_n \cdot \sin(j\theta)\sin(n-j)\theta > 0 \qquad j = 1, \ldots, n-1. \tag{3.2}$$

Condition (3.1) may be rewritten equivalently as

$$v_n = \exp\left(\pi \mathrm{i}\left[\frac{n-1}{p_0}\right]\right)$$

and (3.2) as

$$\left[\frac{j}{p_0}\right] + \left[\frac{n-j}{p_0}\right] = \left[\frac{n-1}{p_0}\right] \qquad j = 1, \ldots, n-1. \tag{3.3}$$

The set of integer numbers n satisfying (3.3) for fixed $p_0 \in \mathbb{R}$ may be described by the following construction (see, e.g., [TS], [KR1], [KR2]).

Let us define a sequence of real numbers p_i and sequences of integer numbers $\nu_i,\ m_i,\ y_i$:

$$p_0 = \frac{\pi}{\theta},\ p_1 = 1,\ \nu_i = \left[\frac{p_i}{p_{i+1}}\right],\ p_{i+1} = p_{i-1} - \nu_{i-1} p_i \qquad i = 1, 2, \ldots \tag{3.4}$$

$$y_{-1} = 0,\ y_0 = 1,\ y_1 = \nu_0,\ y_{i+1} = y_{i-1} + \nu_i y_i \qquad i = 0, 1, 2, \ldots \tag{3.5}$$

$$m_0 = 0, m_1 = \nu_0, m_{i+1} = m_i + \nu_i \qquad i = 0, 1, 2, \ldots. \tag{3.6}$$

It is clear that integer numbers ν_i define the decomposition of p_0 into a continuous fraction

$$p_0 = [\nu_0, \nu_1, \nu_2 \ldots].$$

Let us define a piecewise linear function $n_t,\ t \geqslant 1$

$$n_t = y_{i-1} + (t - m_i) y_i \qquad \text{if } m_i \leqslant t < m_{i+1}.$$

Then for any integer $n > 1$ there exists the unique rational number t such that $n = n_t$.

Lemma 3.1. [KR1]. The integer number $n > 1$ satisfies (3.3) if and only if there exists an integer number t such that $n = n_t$.

We have two types of length 1 strings:

x_α^1 with parity $v_1 = +1$

$x_\alpha^{m_1}$ with parity $v_{m_1} = -1$.

All others strings have a length $n = n_j$, for some integer j, and parity

$$v_j = v_{n_j} = \exp\left(\pi \mathrm{i}\left[\frac{n_j - 1}{p_0}\right]\right).$$

Let us assume that all spins s_i have the following form:

$$2s_i = n_{\chi_i} - 1 \qquad \chi_i \in \mathbb{Z}_+. \tag{3.7}$$

From the assumptions (3.1), (3.3) and (3.7) about spins, length and parity, a simple expression for the sums $\theta_{n,m}(\infty)$ and $\Phi_{n,2s}(\infty)$ follows.

In our paper we consider a special case of rational p_0. The case of irrational p_0 may be obtained as a limit. So, we assume that $p_0 = u/v \in \mathbb{Q}$, $p_0 = [\nu_0, \ldots, \nu_\alpha]$, $\nu_0 \geqslant 2$, $\nu_\alpha \geqslant 2$. Furthermore, we assume that all strings have a length not greater than u (see [TS]). Therefore, for numbers p_i, ν_i, y_i, m_i (see (3.4)–(3.6)), it is enough to keep only the indices $i \leqslant \alpha + 1$. We have also

$$p_{\alpha+1} = \frac{1}{y_\alpha} \qquad p_0 = \frac{y_{\alpha+1}}{y_\alpha} \qquad \text{and} \qquad \mathrm{GCD}\ (y_\alpha, y_{\alpha+1}) = 1.$$

Now we will state the results of calculations for the sums $\theta_{n,m}(\infty)$ and $\Phi_{n,m}(\infty)$. Let us introduce

$$\begin{aligned}
&q_j = (-1)^i (p_i - (j - m_i) p_{i+1}) \qquad \text{if } m_i \leqslant j < m_{i+1} \\
&r(j) = i \qquad \text{if } m_i \leqslant j < m_{i+1} \\
&b_{jk} = \frac{(-1)^{i-1}}{p_0}(q_k n_j - q_j n_k) \qquad \text{if } n_j < n_k \\
&b_{j,m_{i+1}} = 1 \qquad m_i < j < m_{i+1} \\
&\theta_{j,k} = \theta_{n_j,n_k}(\infty) \qquad \Phi_{j,2s} = \Phi_{n_j,2s}(\infty).
\end{aligned} \tag{3.8}$$

Theorem 3.2. (Calculation of the sums $\theta_{j,k}$, $\Phi_{j,2s}$) [KR1].

(1) If $k > j$, and $(j,k) \neq (m_{\alpha+1} - 1, m_{\alpha+1})$ then

$$\theta_{jk} = -n_j \frac{q_k}{p_0}.$$

(2) If $j = m_{\alpha+1} - 1, k = m_{\alpha+1}$, then

$$\theta_{jk} = -n_j \frac{q_k}{p_0} + \frac{(-1)^{\alpha+1}}{2} = (-1)^{\alpha} \frac{p_0 - 2}{2p_0}.$$

(3) If $1 \leqslant j \leqslant m_{\alpha+1}$, then

$$\theta_{jj} = -n_j \frac{q_j}{p_0} + \frac{(-1)^{r(j)}}{2}.$$

(4) (Symmetry). For all $1 \leqslant j, k \leqslant m_{\alpha+1}$

$$\theta_{jk} = \theta_{kj}.$$

(5) If $2s = n_\chi - 1$, then

$$\Phi_{k,2s} = \begin{cases} \dfrac{1}{2p_0}(q_k - q_k n_\chi) & \text{if } n_k > 2s \\ \dfrac{1}{2p_0}(q_k - q_\chi n_k) + \dfrac{(-1)^{r(k)-1}}{2} & \text{if } n_k \leqslant 2s. \end{cases}$$

Note that if $p_0 = \nu_0$ is an integer then

$$2\Phi_{k,2s} = \begin{cases} \dfrac{2sk}{p_0} - \min(k, 2s) & \text{if } 1 \leqslant k, 2s+1 < \nu_0 \\ 0 & \text{if } 1 \leqslant k \leqslant \nu_0, 2s+1 > \nu_0. \end{cases}$$

Now we are going to calculate the vacancy numbers. By definition the vacancy numbers are equal to

$$P_{n_j}(\lambda) = 2Q_{\max}^{n_j} - \lambda_j + 1$$

where

$$Q_{\max}^{n_j} = (-1)^{i-1}\left(Q_\infty^{n_j} - \theta_{jj} - \frac{n_j}{2}\left\{\frac{\sum 2s_m N_m - 2l}{p_0}\right\}\right) - \frac{1}{2} \qquad m_i \leqslant j < m_{i+1}$$

and $\{x\}$ is the fractional part of the real number x.

Here we put

$$Q_\infty^{n_j} = \sum_k \Phi_{n_j,2s_k} N_k - \sum_k \theta_{n_j n_k} \lambda_k + \theta_{n_j n_j}.$$

Let us say a few words about our definition of the vacancy numbers P_{n_j}. In contrast with the XXX model situation, it happens that the vector $x = (\infty, \ldots, \infty)$ for the XXZ case does not appear to be a formal solution to the Bethe equations (2.1). Another difficulty appears in finding a correct boundary for quantum numbers Q_α^n (see (2.6)). A natural boundary is $Q_\infty^{n_j}$ but this number does not appear to be an integer or half-integer one in general. Our choice is based on the attempt to have a combinatorial completeness of Bethe's states and some analytical considerations. In the following we will use the notation $P_j(\lambda)$, Q_∞^j, $Q_{\max}^j, \ldots$ instead of $P_{n_j}(\lambda)$, $Q_\infty^{n_j}$, $Q_{\max}^{n_j}, \ldots$.

After tedious calculations one can find

$$\begin{aligned} &P_j(\lambda) = a_j + 2\sum_{k>j} b_{jk}\lambda_k \qquad j \neq m_{\alpha+1}-1, m_{\alpha+1} \\ &P_{m_{\alpha+1}-1}(\lambda) = a_{m_{\alpha+1}-1} + \lambda_{m_{\alpha+1}} \\ &P_{m_{\alpha+1}}(\lambda) = a_{m_{\alpha+1}} + \lambda_{m_{\alpha+1}-1} \end{aligned} \tag{3.9}$$

where

$$a_j = (-1)^{i-1}\left(\sum_m 2\Phi_{j,2s_m}\cdot N_m + \frac{2lq_j}{p_0} - n_j\left\{\frac{\sum 2s_m N_m - 2l}{p_0}\right\}\right)$$

and b_{jk} for $n_j < n_k$ are defined in (3.8).

From the string conjecture (see [TS], [KR2]) it follows that the number of Bethe's vectors with configuration $\{\lambda_k\}$ is equal to

$$Z(N, s|\{\lambda_k\}) = \prod_j \binom{P_j(\lambda)+\lambda_j}{\lambda_j}.$$

The number of Bethe's vectors with fixed l is equal to

$$Z(N, s|l) = \sum_{\{\lambda_k\}} Z(N, s|\{\lambda_k\}) \tag{3.10}$$

where summation is taken over all configurations $\{\lambda_k\}$, such that $\lambda_k \geqslant 0$, and

$$\sum_{k=1}^{m_{\alpha+1}} n_k\lambda_k = l. \tag{3.11}$$

So, the total number of Bethe's vectors is equal to

$$Z = Z(N, s) = \sum_l Z(N, s|l) \tag{3.12}$$

where we assume that

$$Z(N, s|l) := Z(N, s|\sum 2s_m N_m - l) \qquad \text{for } l \geqslant \sum s_m N_m.$$

The conjecture about combinatorial completeness of Bethe's states for the XXZ model means that

$$Z = \prod_m (2s_m + 1)^{N_m}. \tag{3.13}$$

4. The main combinatorial identity

Let $a_0 = 0, a_1, a_2, \ldots, a_{m_{\alpha+1}}$ be a sequence of real numbers. Then we shall define inductively a sequence $b_2, \ldots, b_{m_1-1}, b_{m_1+1}, \ldots, b_{m_{\alpha+1}}, b_{m_{\alpha+2}}$ by the following rules:

$$\begin{aligned} b_k &= 2a_{k-1} - a_{k-2} - a_k \qquad \text{if } k \neq m_i, k \geqslant 2 \\ b_{m_{i+1}} &= 2a_{m_i-1} - a_{m_i-2} - a_{m_i+1} \qquad \text{if } 1 \leqslant i \leqslant \alpha \\ b_{m_{\alpha+2}} &= a_{m_{\alpha+1}-1} - a_{m_{\alpha+1}-2} + a_{m_{\alpha+1}}. \end{aligned}$$

Then one can check that the converse formulae are

$$a_j = (-1)^{r(j)} \left(\frac{n_j}{p_0} q_{m_{\alpha+1}} (a_{m-1} - a_m) - 2 \sum_k \Phi_{jk} \cdot b_k \right)$$

where Φ_{jk} were defined in (2.5).

For a given configuration $\{\lambda_n\} = \lambda$ let us define the vacancy numbers

$$\begin{aligned} P_j(\lambda) &= a_j + 2 \sum_{k>j} b_{jk} \lambda_k \qquad j \neq m_{\alpha+1} - 1, m_{\alpha+1} \\ P_{m_{\alpha+1}-1}(\lambda) &= a_{m_{\alpha+1}-1} + \lambda_{m_{\alpha+1}} \\ P_{m_{\alpha+1}}(\lambda) &= a_{m_{\alpha+1}} + \lambda_{m_{\alpha+1}-1}. \end{aligned}$$

Let us put

$$Z(\{a_k\}|l) = \sum_{\{\lambda_k\}} \prod_{k=1}^{m_{\alpha+1}} \binom{P_k(\lambda) + \lambda_k}{\lambda_k}$$

where summation is taken over all configurations $\{\lambda_k\}$ such that

$$\sum_{k=1}^{m} n_k \lambda_k = l.$$

Note that a binomial coefficient $\binom{\alpha}{\nu}$ for real α and integer positive ν is defined as

$$\binom{\alpha}{\nu} = \frac{\alpha(\alpha-1)\ldots(\alpha-\nu+1)}{\nu!}.$$

Theorem 4.1. (The main combinatorial identity.) We have

$$Z(\{a_k\}|l) = \mathrm{Res}_{u=0} f(u) u^{-l-1}\, \mathrm{d}u$$

where

$$f(u) = (1+u)^{2l+2a_1-a_2} \prod_{k \neq m_i} \left(\frac{1-u^{n_k}}{1-u} \right)^{2a_{k-1}-a_k-a_{k-2}} \cdot \prod_{i=1}^{\alpha} \left(\frac{1-u^{y_i}}{1-u} \right)^{2a_{m_i-1}-a_{m_i-2}-a_{m_{i+1}}} \left(\frac{1-u^{y_{\alpha+1}}}{1-u} \right)^{a_{m_{\alpha+1}}+a_{m_{\alpha+1}-1}-a_{m_{\alpha+1}-2}}.$$

Proof. We shall divide the proof into a few steps.

Step I. Let us put $m_{\alpha+1} = m$. We define a sequence of formal power series $\varphi_1, \ldots, \varphi_m$ in variables $z_1, \ldots, z_m, z_0$ by the following rules:

$$\varphi_m(z_m) = (1 - z_m)^{-(a_m+1)}(1 - z_0(1 - z_m)^{-1})^{-1}$$
$$\varphi_{m-1}(z_{m-1}, z_m) = (1 - z_{m-1})^{-(a_{m-1}+1)}\varphi_m((1 - z_{m-1})^{-1}z_m)$$
$$\vdots$$
$$\varphi_k(z_k, \ldots, z_m) = (1 - z_k)^{-(a_k+1)}\varphi_{k+1}((1 - z_k)^{-2b_{k,k+1}} \times z_{k+1}, \ldots, (1 - z_k)^{-2b_{k,l}} z_l, \ldots, (1 - z_k)^{-2b_{k,m}} z_m)$$
$$\vdots$$
$$\varphi_1(z_1, \ldots, z_m) = (1 - z_1)^{-(a_1+1)}\varphi_2((1 - z_1)^{-2b_{1,2}} z_2, \ldots, (1 - z_1)^{-2b_{1,l}} \times z_l, \ldots, (1 - z_1)^{-2b_{1,m}} z_m).$$

Lemma 4.2. In the power series $\varphi_1(z_1, \ldots, z_m)$ a coefficient before $z_o^{\nu_0} z_1^{\nu_1} \ldots z_m^{\nu_m}$ is equal to

$$\prod_{j=1}^{m-1} \binom{P_j(\nu) + \nu_j}{\nu_j} \cdot \binom{a_m + \nu_m + \nu_0}{\nu_m}.$$

Proof.

$$\varphi_m(z_m) = \sum_{\nu_0,\nu_m} z_0^{\nu_0} z_m^{\nu_m} \binom{a_m + \nu_m + \nu_0}{\nu_m}.$$

Let us assume that

$$\varphi_k(z_k, \ldots, z_m) = \sum_{\nu_0,\nu_k,\ldots,\nu_m} A_k(\nu_k, \ldots, \nu_m; \nu_0) z_0^{\nu_0} z_k^{\nu_k} \ldots z_m^{\nu_m}$$

then

$$\begin{aligned}\varphi_{k-1}&(z_{k-1}, \ldots, z_m)\\ &= (1 - z_{k-1})^{-(a_{k-1}+1)}\varphi_k((1 - z_k)^{-2b_{k,k+1}} z_{k+1}, \ldots, (1 - z_k)^{-2b_{k,m}} z_m)\\ &= \sum_{\nu_0,\nu_k,\ldots,\nu_m} A_k(\nu_k, \ldots, \nu_m; \nu_0)(1 - z_{k-1})^{-(p_{k-1}(\nu)+1)} z_0^{\nu_0} z_k^{\nu_k} \ldots z_m^{\nu_m}\\ &= \sum_{\nu_0,\nu_{k-1},\ldots,\nu_m} A_k(\nu_k, \ldots, \nu_m; \nu_0) \binom{P_{k-1}(\nu) + \nu_{k-1}}{\nu_{k-1}} z_0^{\nu_0} z_{k-1}^{\nu_{k-1}} \ldots z_m^{\nu_m}.\end{aligned}$$

Consequently,

$$A_{k-1}(\nu_{k-1}, \nu_k, \ldots, \nu_m; \nu_0) = A_k(\nu_k, \ldots, \nu_m; \nu_0) \cdot (P_{k-1}(\nu) + \nu_{k-1}\nu_{k-1}).$$

□

From lemma 4.2 it follows that the sum $Z(\{a\}|l)$ is equal to the coefficient before t^l in the power series of $\psi(z, t)$, which has been obtained from $\varphi_1(z_1, \ldots, z_m)$ after substitution

$$z_j = t^{n_j} \qquad j \neq m - 1$$
$$z_{m-1} = t^{n_{m-1}} z_0^{-1}.$$

Step II. Calculation of the power series for $\psi(z, t)$. Let us define

$$z_k^{(l)} := (1 - z_l^{(l-1)})^{-2b_{l,k}} \cdot z_k^{(l-1)} \qquad l \geqslant 1 \tag{4.1}$$
$$z_k^{(0)} = t^{n_k} \qquad \text{if } k \neq m - 1 \text{ and } z_{m-1}^{(0)} = t^{n_{m-1}} z_0^{-1}.$$

Then we have

$$\begin{aligned}\varphi_1(z_1,\ldots,z_m) &= (1-z_1)^{-(a_1+1)}\varphi_2(z_2^{(1)},z_3^{(1)},\ldots,z_m^{(1)})\\ &= (1-z_1)^{-(a_1+1)}(1-z_2^{(1)})^{-(a_2+1)}\varphi_3(z_3^{(2)},z_4^{(2)},\ldots,z_m^{(2)})\\ &\;\;\vdots\\ &= \prod_{j=1}^{m-1}(1-z_j^{(j-1)})^{-(a_j+1)}\cdot\varphi_{m-1}(z_{m-1}^{(m-2)},z_m^{(m-2)}).\end{aligned} \tag{4.2}$$

In order to compute a formal series $z_k^{(l)}$, we define (see, e.g., [K1]) a sequence of polynomials $Q_m(t)$ using the following recurrence relation:

$$Q_{m+1}(t) = Q_m(t) - tQ_{m-1}(t) \qquad m \geqslant 0$$
$$Q_0(t) = Q_{-1}(t) := 1.$$

Lemma 4.3. (Formulae for power series $z_k^{(l)}$.) Let us assume that $m_i \leqslant k < m_{i+1}$ and put $m_0 := 1$. Then we have ($Q_k := Q_k(t)$)

(1) $z_k^{(k-1)} = Q_{k-1}^{-2}Q_{m_i-2}z_k^{(0)}$.

(2) $1 - z_k^{(k-1)} = Q_kQ_{k-1}^{-2}Q_{k-2}$, if $k \neq m_i$.

(3) If $k = m_i$, $i \geqslant 1$, then $1 - z_k^{(k-1)} = Q_kQ_{k-1}^{-2}Q_{m_{i-1}-2}$.

(4) After specialization $t := u/(1+u)^2$ one can find (note that $m_i \leqslant k < m_{i+1}$)

$$Q_k(u) = 1 - \frac{1-u^{n_k+2y_i}}{(1-u)(1+u)^{n_k+2y_i-1}}.$$

(5) If $k \neq m_i + 1$ and $m_i \leqslant k < m_{i+1}$, then

$$z_k^{(k-2)} = Q_{k-3}^2Q_{k-2}^{-4}Q_{m_i-2}^2z_k^{(0)}.$$

Proof. This follows by induction from (4.1) and the properties of polynomials $Q_k(t)$ (compare [K1], lemma 2). □

Corollary 4.4. (1)

$$z_m^{(m-2)} = Q_{m-3}^2Q_{m-2}^{-2}t^{n_m} \qquad z_{m-1}^{(m-2)} = Q_{m-2}^{-2}Q_{m_{\alpha-2}}^2t^{n_{m-1}}z_0^{-1}.$$

(2) Let us denote by $\varphi_{m-1}(u, z_0)$ a specialization $t = u/(1-u)^2$ of formal series $\varphi_{m-1}(z_{m-1}^{(m-2)}, z_m^{(m-1)})$ and let $\varphi_{m-1}(u)$ be a constant term of series $\varphi_{m-1}(u, z_0)$ with respect to variable z_0. Then

$$\varphi_{m-1}(u) = (1-u^{y_{\alpha+1}})^{a_m+a_{m-1}+1}(1-u^{y_{\alpha+1}-y_\alpha})^{-(a_{m-1}+1)}(1-u^{y_\alpha})^{-(a_m+1)}.$$

Note that $m = m_{\alpha+1}$.

Step III. Combining (4.2), lemma 4.3 and corollary 4.4 after some simplifications we obtain a proof of theorem 4.1. □

Corollary 4.5. (Combinatorial completeness of Bethe's states for XXZ model of arbitrary spins.)

$$Z = \prod_m (2s_m + 1)^{N_m}. \tag{4.3}$$

Examples below give an illustration to our result about completeness of Bethe's states for the spin-$\frac{1}{2}$ XXZ model (examples 1 and 3) and for the spin-1 XXZ model (example 4).

Example 1. We compute firstly the quantities q_j, a_j (see (3.8)) and after this consider a numerical example. From (3.4)–(3.6) and (3.8) it follows that

$$q_j = (-1)^i \frac{p_0 - n_j p_{i+1}}{y_i}.$$

Using theorem 3.2(5) we obtain (see (3.9))

$$\begin{aligned} a_j = (-1)^{i-1} n_j \left[\frac{\sum 2s_m N_m - 2l}{p_0}\right] + (-1)^i (n_j + q_j) \left(\frac{\sum 2s_m N_m - 2l}{p_0}\right) \\ + \frac{n_j}{p_0} \sum_{\{m: 2s_m \leqslant n_j\}} N_m \left(\frac{p_{i+1}}{y_i}(2s_m + 1) + (-1)^i q_\chi\right) \\ + \sum_{\{m: 2s_m \leqslant n_j\}} N_m \left(1 - \frac{1}{y_i}(2s_m + 1)\right). \end{aligned} \tag{4.4}$$

Let us consider the case when all spins are equal to $\frac{1}{2}$ and let N be the number of spins, then

(i) $0 \leqslant j < m_1 (= \nu_0)$. Then $r(j) = i = 0$ and $n_j = j, q_j = p_0 - j$,

$$a_j = -n_j \left[\frac{N - 2l}{p_0}\right] + N - 2l + \delta_{n_j,1} \frac{N}{p_0}(2 - p_0 + q_\chi).$$

(ii) $m_1 \leqslant j < m_2 (= \nu_0 + \nu_1)$. Then $r(j) = 1$ and $n_j = 1 + (j - m_1)\nu_0$, $q_j = (p_0 - \nu_0)(j - m_1) - 1$,

$$a_j = n_j \left[\frac{N - 2l}{p_0}\right] - \frac{N - 2l}{\nu_0}(n_j - 1) - \delta_{n_j,1} \frac{N}{p_0}(2 - p_0 + q_\chi).$$

For example,

$$a_{m_1} = \left[\frac{N - 2l}{p_0}\right] - \frac{N}{p_0}(2 - p_0 + q_\chi).$$

(iii) $m_2 \leqslant j < m_3$ $(= \nu_0 + \nu_1 + \nu_2)$. Then $r(j) = 2$ and $n_j = \nu_0 + (j - m_2)(1 + \nu_0 \nu_1)$, $q_j = p_0 - \nu_0 - (j - m_2)(1 - \nu_1(p_0 - \nu_0))$

$$a_j = -n_j \left[\frac{N - 2l}{p_0}\right] + \frac{N - 2l}{\nu_0 + (1/\nu_1)} \left(n_j + \frac{1}{\nu_1}\right).$$

Consequently,

$$a_{m_2} = -\nu_0 \left[\frac{N - 2l}{p_0}\right] + (N - 2l).$$

Now let us assume $p_0 = 3 + \frac{1}{3}, N = 5$. It is clear that in our case $\chi = 2$ (see (3.7)) and $q_\chi = p_0 - 2$. Below we give all solutions $\lambda = \{\lambda_1, \lambda_2, \ldots\}$ to the equation (3.11) when $0 \leqslant l \leqslant 2$ and compute the corresponding vacancy numbers $P_j = P_j(\lambda)$ (see (3.9)) and number of states $Z = Z(N, \frac{1}{2}|\{\lambda_k\})$ (see (3.10) and (3.12)):

$$\begin{array}{lllll}
l=0 & \{0\} & P_j = 0 & Z = 1 & \\
 & & & & Z(5, \tfrac{1}{2}|0) = 1 \\
l=1 & \{1,0,0\} & P_1 = 3 & Z = 4 & \\
 & \{0,0,1\} & P_3 = 0 & Z = 1 & \\
 & & & & Z(5, \tfrac{1}{2}|1) = 5 \\
l=2 & \{0,1,0\} & P_2 = 1 & Z = 2 & \\
 & \{2,0,0\} & P_1 = 1 & Z = 3 & \\
 & \{0,0,2\} & P_3 = 0 & Z = 1 & \\
 & \{1,0,1\} & \begin{cases} P_1 = 3 \\ P_3 = 0 \end{cases} & Z = 4 & \\
 & & & & Z(5, \tfrac{1}{2}|2) = 10.
\end{array}$$

Consequently,

$$Z(N = 5, \tfrac{1}{2}) = 2(1 + 5 + 10) = 32 = 2^5.$$

Note that our formula (3.10) for the number of Bethe's states with fixed spin l, namely $Z(N, s|l)$, works for $l \geqslant \sum s_m N_m$ as well as for small $l \leqslant \sum s_m N_m$.

In the appendix we consider two additional examples, one when all spins are equal to $\frac{1}{2}$, another when all spins are equal to 1. The last example seems to be interesting because a non-admissible configuration appears.

Remark 1. It is easy to see that for fixed l and sufficiently big $N = \sum 2s_m N_m$ all vacancy numbers $P_j(\lambda)$ are non-negative. This is not the case for particular N and we must consider really the configurations with

$$P_j(\lambda) + \lambda_j < 0 \qquad \text{for some } j \tag{4.5}$$

in order to have a correct answer for $Z^{XXZ}(N, s|l)$. See the appendix, example 4, $l = 4$, (♣). Let us note that for the XXX model the non-admissible configurations (i.e. those satisfying (4.5)) give a zero contribution to the sum $Z^{XXX}(N, s|l)$ [K2].

Remark 2. One can rewrite the expressions (3.9) for vacancy numbers in the following form if $m_i \leqslant j < m_{i+1}$,

$$\begin{aligned} P_j(\lambda) = (-1)^{i-1}\left(\sum_m 2\Phi_{j,2s_m} \cdot N_m - n_j\left\{\frac{\sum 2s_m N_m - 2l}{p_0}\right\}\right) \\ - \sum_k 2(-1)^{r(k)}\widetilde{\theta}_{jk}\lambda_k - \delta_{j,m_{\alpha+1}-1}\lambda_{m_{\alpha+1}} + \delta_{j,m_{\alpha+1}}\lambda_{m_{\alpha+1}-1} \end{aligned}$$

where $\widetilde{\theta}_{jk} = (-1)^{r(j)+r(k)} n_j q_k / p_0$, if $j \leqslant k$ and $\widetilde{\theta}_{jk} = \widetilde{\theta}_{kj}$.

Let us introduce the symmetric matrix $\Theta = (\widetilde{\theta}_{ij})_{1 \leqslant i,j \leqslant m_{\alpha+1}}$. We can find the inverse matrix $\Theta^{-1} := (c_{ij})$ and compute its determinant.

Theorem 4.6. Matrix elements c_{ij} of the inverse matrix Θ^{-1} are given by the following rules:

(i) $c_{ij} = c_{ji}$ and $c_{ij} = 0$, if $|i - j| \geqslant 2$;
(ii) $c_{j-1,j} = (-1)^{i-1}$, if $m_i \leqslant j < m_{i+1}$;

(iii)

$$c_{jj} = \begin{cases} 2(-1)^i & \text{if } m_i \leqslant j < m_{i+1} - 1, i \leqslant \alpha \\ (-1)^i & \text{if } j = m_{i+1} - 1, i \leqslant \alpha \\ (-1)^{\alpha+1} & \text{if } j = m_{\alpha+1}. \end{cases}$$

Theorem 4.7. We have

$$\det|\Theta^{-1}| = y_{\alpha+1}.$$

The proofs of theorems 4.6 and 4.7 follow from [KR2], the appendix, and relations

$$y_i p_i + y_{i-1} p_{i+1} = p_0 \qquad 0 \leqslant i \leqslant \alpha + 1.$$

Example 2. For $p_0 = 4 + \frac{1}{5}$ using theorem 4.6 one can find

$$\Theta^{-1} = \begin{pmatrix} 2 & -1 & & & & & & & \\ -1 & 2 & -1 & & & & & & \\ & -1 & 1 & 1 & & & & & \\ & & 1 & -2 & 1 & & & & \\ & & & 1 & -2 & 1 & & & \\ & & & & 1 & -2 & 1 & & \\ & & & & & 1 & -2 & 1 & \\ & & & & & & 1 & -1 & -1 \\ & & & & & & & -1 & 1 \end{pmatrix}.$$

5. Conclusion

In this paper we have proved a very general combinatorial identity (theorem 4.1). As a particular case we have proved a combinatorial completeness of Bethe's states for the generalized XXZ model (corollary 4.5). One can construct a natural q-analogue for the number of Bethe's states with fixed spin l (see (3.10)). Namely, let us consider a vector

$$\widetilde{\lambda} = (\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_{m_{\alpha+1}})$$

where $\widetilde{\lambda}_j = (-1)^{r(j)} \lambda_j$ and a matrix $E = (e_{jk})_{1 \leqslant j,k \leqslant m_{\alpha+1}}$, where

$$e_{jk} = (-1)^{r(k)} (\delta_{j,k} - \delta_{j,m_{\alpha+1}-1} \cdot \delta_{k,m_{\alpha+1}} + \delta_{j,m_{\alpha+1}} \cdot \delta_{k,m_{\alpha+1}-1}).$$

Then it is easy to check that

$$P_j(\lambda) + \lambda_j = ((E - 2\Theta)\widetilde{\lambda}^t + b^t)_j$$

where $b = (b_1, \ldots, b_{m_{\alpha+1}})$ and

$$b_j = (-1)^{r(j)} \left(n_j \left\{ \frac{\sum 2 s_m N_m - 2l}{p_0} \right\} - \sum_m 2\Phi_{j,2s_m} \cdot N_m \right).$$

We consider the following q-analogue of (3.10),

$$\sum_\lambda q^{\frac{1}{2}\widetilde{\lambda} B \widetilde{\lambda}^t} \prod_j \begin{bmatrix} ((E - B)\widetilde{\lambda}^t + b^t)_j \\ \lambda_j \end{bmatrix}_q \tag{5.1}$$

where summation is taken over all configurations $\lambda = \{\lambda_k\}$ such that

$$\sum_{k=1}^{m_{\alpha+1}} n_k \lambda_k = l \qquad \lambda_k \geqslant 0 \qquad \text{and} \qquad B = 2\Theta.$$

The thermodynamical limit of (5.1) (i.e. $N_m \to \infty$) comes to

$$\sum_{\lambda} \frac{q^{\frac{1}{2}\widetilde{\lambda} B \widetilde{\lambda}^t}}{\prod_j (q)_{\lambda_j}}. \tag{5.2}$$

Summation in (5.2) is the same as in (5.1) and $(q)_n := (1-q)\cdots(1-q^n)$. Here $B = C_1 \otimes \Theta$ and $C_1 = (2)$ is the Cartan matrix of type A_1.

It is an interesting problem to find a representation theory meaning of (5.2), when $B = C_k \otimes \Theta$ and C_k is the Cartan matrix of type A_k.

Another interesting question concerns the degeneration of Bethe's states for the XXZ model into those for the XXX one. More exactly, we had proved (see (4.3)) that

$$\prod_m (2s_m + 1)^{N_m} = \sum_{l=0}^{N} Z^{XXZ}(N, s|l) \tag{5.3}$$

where $N = \sum_m 2s_m N_m$ and $Z^{XXZ}(N, s|l)$ is given by (3.10).

On the other hand, it follows from a combinatorial completeness of Bethe's states for the XXX model (see [K1]) that

$$\prod_m (2s_m + 1)^{N_m} = \sum_{l \geqslant 0}^{\frac{1}{2}N} (N - 2l + 1) Z^{XXX}(N, s|l) \tag{5.4}$$

where $Z^{XXX}(N, s|l)$ is the multiplicity of the $(\frac{1}{2}N - l)$-spin irreducible representation $V_{\frac{1}{2}N-l}$ of $sl(2)$ in the tensor product

$$V_{s_1}^{\otimes N_1} \otimes \cdots \otimes V_{s_m}^{\otimes N_m}.$$

It is an interesting question to find a combinatorial proof that

$$\text{RHS}(5.3) = \text{RHS}(5.4).$$

Another interesting task is to compare our results with those obtained in [KM]. We intend to consider these questions and also to study in more detail the case $p_0 = \nu_0$ as an integer and all spins equal to $(\nu_0 - 2)/2$ in separate publications.

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Appendix

Example 3. Using the same notation as in example 1, we consider the case $s = \frac{1}{2}$, $p_0 = 3 + \frac{1}{3}$, $N = 8$ and compute the vacancy numbers $P_j(\lambda)$ and numbers of states $Z = Z(N, \frac{1}{2}|\{\lambda_k\})$:

$l=0$	$\{0\}$	$P_j=0$	$Z=1$	
				$Z(8,\frac{1}{2}\|0)=1$
$l=1$	$\{1,0,0\}$	$P_1=5$	$Z=6$	
	$\{0,0,1\}$	$P_3=1$	$Z=2$	
				$Z(8,\frac{1}{2}\|1)=8$
$l=2$	$\{2,0,0\}$	$P_1=3$	$Z=10$	
	$\{0,0,2\}$	$P_3=1$	$Z=3$	
	$\{0,1,0\}$	$P_2=2$	$Z=3$	
	$\{1,0,1\}$	$\begin{cases}P_1=5\\P_3=1\end{cases}$	$Z=12$	
				$Z(8,\frac{1}{2}\|2)=28$
$l=3$	$\{3,0,0\}$	$P_1=2$	$Z=10$	
	$\{0,0,3\}$	$P_3=0$	$Z=1$	
	$\{0,0,0,0,0,1\}$	$P_6=2$	$Z=3$	
	$\{1,1,0\}$	$\begin{cases}P_1=4\\P_2=2\end{cases}$	$Z=15$	
	$\{0,1,1\}$	$\begin{cases}P_2=4\\P_3=0\end{cases}$	$Z=5$	
	$\{2,0,1\}$	$\begin{cases}P_1=4\\P_3=0\end{cases}$	$Z=15$	
	$\{1,0,2\}$	$\begin{cases}P_1=6\\P_3=0\end{cases}$	$Z=7$	
				$Z(8,\frac{1}{2}\|3)=56$
$l=4$	$\{4,0,0\}$	$P_1=0$	$Z=1$	
	$\{0,0,4\}$	$P_3=0$	$Z=1$	
	$\{0,2,0\}$	$P_2=0$	$Z=1$	
	$\{0,0,0,1\}$	$P_4=0$	$Z=1$	
	$\{2,1,0\}$	$\begin{cases}P_1=2\\P_2=0\end{cases}$	$Z=6$	
	$\{0,1,2\}$	$\begin{cases}P_2=4\\P_3=0\end{cases}$	$Z=5$	
	$\{3,0,1\}$	$\begin{cases}P_1=2\\P_3=0\end{cases}$	$Z=10$	
	$\{1,0,3\}$	$\begin{cases}P_1=6\\P_3=0\end{cases}$	$Z=7$	
	$\{2,0,2\}$	$\begin{cases}P_1=4\\P_3=0\end{cases}$	$Z=15$	
	$\{1,0,0,0,0,1\}$	$\begin{cases}P_1=4\\P_6=0\end{cases}$	$Z=5$	
	$\{0,0,1,0,0,1\}$	$\begin{cases}P_3=2\\P_6=0\end{cases}$	$Z=3$	

$$\{1,1,1,0,0,0\} \quad \begin{cases} P_1 = 4 \\ P_2 = 2 \quad Z = 15 \\ P_3 = 0 \end{cases}$$

$$Z(8, \tfrac{1}{2}|4) = 70.$$

Consequently,

$$Z(N = 8, \tfrac{1}{2}|l) = \binom{8}{l} \qquad 0 \leqslant l \leqslant 4$$

and

$$Z(N = 8, \tfrac{1}{2}) = 2(1 + 8 + 28 + 56) + 70 = 256 = 2^8.$$

Example 4. Let us consider the case when all spins are equal to 1 and let N be the number of spins. We compute firstly the quantities a_j (see (3.8)) and after this consider a numerical example.

(i) $0 \leqslant j < m_1 (= \nu_0)$. Then $r(j) = i = 0$ and $n_j = j, q_j = p_0 - j$,

$$a_j = \begin{cases} -j\left[\dfrac{2N - 2l}{p_0}\right] + 2N - 2l & \text{if } j > 2 \\ -j\left[\dfrac{2N - 2l}{p_0}\right] + \dfrac{jN}{p_0}(3 + q_\chi) - 2l & \text{if } j \leqslant 2. \end{cases}$$

(ii) $m_1 \leqslant j < m_2 (= \nu_0 + \nu_1)$. Then $r(j) = 1$ and $n_j = 1 + (j - m_1)\nu_0$, $q_j = (p_0 - \nu_0)(j - m_1) - 1$,

$$a_j = n_j\left[\frac{2N - 2l}{p_0}\right] - \frac{2N - 2l}{\nu_0}(n_j - 1) - \delta_{n_j,1}\frac{N}{p_0}(3 - p_0 + q_\chi).$$

(iii) $m_2 \leqslant j < m_3 (= \nu_0 + \nu_1 + \nu_2)$. Then $r(j) = 2$ and $n_j = \nu_0 + (j - m_2)(1 + \nu_0\nu_1)$, $q_j = p_0 - \nu_0 - (j - m_2)(1 - \nu_1(p_0 - \nu_0))$,

$$a_{m_2} = -\nu_0\left[\frac{2N - 2l}{p_0}\right] + 2N - 2l.$$

Now let us assume $p_0 = 3 + \frac{1}{3}, N = 5$. It is clear that $\chi = 6$ and $q_\chi = \frac{1}{3}$. Below we give all solutions $\lambda = \{\lambda_1, \lambda_2, \ldots\}$ to the equation (3.11) when $0 \leqslant l \leqslant 5$ and compute the corresponding vacancy numbers $P_j = P_j(\lambda)$ (see (3.9)) and number of states $Z = Z(N, 1|\{\lambda_k\})$ (see (3.10) and (3.12)):

$$l = 0 \quad \{0\} \quad P_j = 0 \quad Z = 1$$

$$Z(5, 1|0) = 1$$

$$l = 1 \quad \{1,0,0\} \quad P_1 = 1 \quad Z = 2$$

$$\{0,0,1\} \quad P_3 = 2 \quad Z = 3$$

$$Z(5, 1|1) = 5$$

$$l = 2 \quad \{2,0,0\} \quad P_1 = 0 \quad Z = 1$$

$$\{0,0,2\} \quad P_3 = 1 \quad Z = 3$$

$$\{0,1,0\} \quad P_2 = 4 \quad Z = 5$$

$$\{1,0,1\} \quad \begin{cases} P_1 = 2 \\ P_3 = 1 \end{cases} \quad Z = 6$$

$$Z(5, 1|2) = 15$$

$l=3$	$\{3,0,0\}$	$P_1=-2$	$Z=0$	
	$\{0,0,3\}$	$P_3=1$	$Z=4$	
	$\{0,0,0,0,0,1\}$	$P_6=1$	$Z=2$	
	$\{1,1,0\}$	$\begin{cases} P_1=0 \\ P_2=2 \end{cases}$	$Z=3$	
	$\{0,1,1\}$	$\begin{cases} P_2=4 \\ P_3=1 \end{cases}$	$Z=10$	
	$\{2,0,1\}$	$\begin{cases} P_1=0 \\ P_3=1 \end{cases}$	$Z=2$	
	$\{1,0,2\}$	$\begin{cases} P_1=2 \\ P_3=1 \end{cases}$	$Z=9$	
				$Z(5,1\|3)=30$
$l=4$	$\{4,0,0\}$	$P_1=-3$	$Z=0$	
	$\{0,0,4\}$	$P_3=0$	$Z=1$	
	$\{0,2,0\}$	$P_2=2$	$Z=6$	
	$\{0,0,0,1\}$	$P_4=-2$	$Z=-1$	(♣)
	$\{2,1,0\}$	$\begin{cases} P_1=-1 \\ P_2=2 \end{cases}$	$Z=0$	
	$\{0,1,2\}$	$\begin{cases} P_2=6 \\ P_3=0 \end{cases}$	$Z=7$	
	$\{3,0,1\}$	$\begin{cases} P_1=-1 \\ P_3=0 \end{cases}$	$Z=0$	
	$\{1,0,3\}$	$\begin{cases} P_1=3 \\ P_3=0 \end{cases}$	$Z=4$	
	$\{2,0,2\}$	$\begin{cases} P_1=1 \\ P_3=0 \end{cases}$	$Z=3$	
	$\{1,0,0,0,0,1\}$	$\begin{cases} P_1=1 \\ P_6=2 \end{cases}$	$Z=6$	
	$\{0,0,1,0,0,1\}$	$\begin{cases} P_3=2 \\ P_6=2 \end{cases}$	$Z=9$	
	$\{1,1,1,0,0,0\}$	$\begin{cases} P_1=1 \\ P_2=4 \\ P_3=0 \end{cases}$	$Z=10$	
				$Z(5,1\|4)=45$
$l=5$	$\{5,0,0\}$	$P_1=-5$	$Z=0$	
	$\{0,0,5\}$	$P_3=0$	$Z=1$	
	$\{4,0,1\}$	$\begin{cases} P_1=-3 \\ P_3=0 \end{cases}$	$Z=0$	
	$\{1,0,4\}$	$\begin{cases} P_1=3 \\ P_3=0 \end{cases}$	$Z=4$	

$$\{3,0,2\} \quad \begin{cases} P_1 = -1 \\ P_3 = 0 \end{cases} \quad Z = 0$$

$$\{2,0,3\} \quad \begin{cases} P_1 = 1 \\ P_3 = 0 \end{cases} \quad Z = 3$$

$$\{3,1,0\} \quad \begin{cases} P_1 = -3 \\ P_2 = 0 \end{cases} \quad Z = 0$$

$$\{0,1,3\} \quad \begin{cases} P_2 = 6 \\ P_3 = 0 \end{cases} \quad Z = 7$$

$$\{1,2,0\} \quad \begin{cases} P_1 = -1 \\ P_2 = 0 \end{cases} \quad Z = 0$$

$$\{0,2,1\} \quad \begin{cases} P_2 = 2 \\ P_3 = 0 \end{cases} \quad Z = 6$$

$$\{1,0,0,1\} \quad \begin{cases} P_1 = 1 \\ P_4 = 0 \end{cases} \quad Z = 2$$

$$\{0,0,1,1\} \quad \begin{cases} P_3 = 2 \\ P_4 = 0 \end{cases} \quad Z = 3$$

$$\{0,1,0,0,0,1\} \quad \begin{cases} P_2 = 2 \\ P_6 = 0 \end{cases} \quad Z = 3$$

$$\{2,0,0,0,0,1\} \quad \begin{cases} P_1 = -1 \\ P_6 = 0 \end{cases} \quad Z = 0$$

$$\{0,0,2,0,0,1\} \quad \begin{cases} P_3 = 2 \\ P_6 = 0 \end{cases} \quad Z = 6$$

$$\{1,0,1,0,0,1\} \quad \begin{cases} P_1 = 1 \\ P_3 = 2 \\ P_6 = 0 \end{cases} \quad Z = 6$$

$$\{2,1,1\} \quad \begin{cases} P_1 = -1 \\ P_2 = 2 \\ P_3 = 0 \end{cases} \quad Z = 0$$

$$\{1,1,2\} \quad \begin{cases} P_1 = 1 \\ P_2 = 4 \\ P_3 = 0 \end{cases} \quad Z = 10$$

$$Z(5,1|5) = 51$$

$$Z(N=5,1) = 2(1+5+15+30+45)+51 = 243 = 3^5.$$

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